The finite element method (FEM) is a numerical technique used in finite element analysis (FEA) to study various physical phenomena, including structural and fluid behavior, thermal transport, wave propagation, and biological cell growth. Understanding these processes requires mathematical modeling using partial differential equations (PDEs). To solve these PDEs, computers rely on numerical techniques like FEM, which has been developed over the years. Initially, FEM showed great promise in modeling mechanical applications related to aerospace and civil engineering. Its potential is now being explored in areas such as fluid-structure interaction, thermomechanical problems, biomechanics, biomedical engineering, piezoelectric, ferroelectric, and electromagnetics. While alternative methods have been proposed, their commercial viability remains unproven. FEM is a tool that requires skilled users to produce accurate results. Before applying FEM, it's essential to understand the different types of PDEs and how they relate to FEM. This knowledge is crucial for everyone, regardless of their motivation for using finite element analysis. PDEs can be categorized as elliptic, hyperbolic, or parabolic. Each type requires specific inputs, such as boundary and/or initial conditions. For example, the Poisson equation is an elliptic PDE, while the Wave equation is a hyperbolic one. FEM uses variational approaches based on energy minimization to solve elliptic PDEs. To address the limitations of original FEM technology in solving hyperbolic PDEs, modifications have been developed over time. Given text analysis led to the following discussion: Considering the suitability of numerical frameworks in solving Partial Differential Equations (PDEs), it's crucial to identify well-posed solutions that ensure reliability and accuracy. A poorly chosen framework can lead to "improperly posed" solutions, characterized by large oscillations or limited existence in specific domains or time periods. In contrast, well-posed explications guarantee unique solutions continuously available for defined data. The principle of minimization of energy underlies the finite element method, as it seeks to find configurations that minimize total energy under applied boundary conditions. This fundamental concept ensures consistent results regardless of simulation repetitions. The process of solving differential equations can be simplified by converting them into integral equations. This is achieved by integrating both sides of the equation with respect to a function v(x), resulting in an equation involving the unknown function u(x) and its derivatives. The order of continuity required for u(x) is reduced, making it easier to find a solution. The Riesz representation theorem can be used to prove the existence of a unique solution for u(x). If f(x) is smooth, it ensures that u(x) is also smooth. Discretization is the next step, where the integral equation is converted into a numerical form by dividing the domain into small elements and nodes. At each node, the unknown function u(x) is calculated using interpolation functions, which are often referred to as shape or ansatz functions. These functions allow for the calculation of values inside an element using nodal values. The weak form can be rewritten in terms of these interpolation functions, resulting in a matrix equation that can be solved using well-known theories. The stiffness matrix [K], nodal unknowns {u}, and residual vector {R} are key components of this matrix equation. Numerical integration schemes, such as Gauss or Newton-Cotes quadrature, can be used to handle the integrations involved in forming these matrices. Choosing interpolation functions requires knowledge of functional spaces, including Hilbert and Sobolev spaces. Further details on this topic can be found in guides to learning Finite Element Analysis (FEA). and terms for other equilibrium states between elements. FEM Conclusion We hope this article has covered the answers to your most important questions regarding the finite element method. If you'd like to see it in practice, SimScale offers the possibility to carry out finite element analyses directly within your web browser. To discover all the features provided by the SimScale cloud-based simulation platform, download our overview or watch one of our recent webinars. For a head start with SimScale, check out our blog article "9 Learning Resources to Get You Started with Engineering Simulation". References Schnellback, Probleme der Variationsrechnung, Journal für die reine und Angewandte Mathematik , v. 41, pp. 293-363 (1851)R. Courant, Variational methods for the solution of problems of equilibrium and vibrations, Bulletin of American Mathematical Society, v. 49, pp. 1-23 (1943)M. J. Turner, R. M. Clough, H. C. Martin and L. J. Topp, Stiffness and deflection analysis of complex structures, Journal of Aeronautical Science, v. The finite element method is a powerful tool for solving partial differential equations by breaking down the problem into smaller, manageable parts. In this approach, the rod's length is divided into seven non-uniform segments, each represented by a linear basis function. The function u is approximated using a combination of these basis functions, with coefficients u0 through u7. This method offers great flexibility in selecting elements and basis functions, allowing for uniform or non-uniform distribution along the x-axis. The choice of basis function can be limited to those with narrow support, ensuring that they overlap along the x-axis. Other functions may be used instead of linear ones, depending on the problem at hand. The finite element method's theory is well-developed, providing useful error estimates and bounds for errors when solving numerical model equations. Conservation laws like the law of conservation of energy and momentum can be written as partial differential equations. These equations describe how changes in variables such as temperature, density, and velocity affect each other. A key concept in differential equations is the derivative, which measures how a dependent variable changes with respect to an independent variable. For example, consider a solid with time-varying temperature but no significant changes in space. In this case, the equation for conservation of internal energy can be expressed as a simple change in temperature over time due to a heat source g: Here, denotes density and Cp is heat capacity. Temperature T is the dependent variable and time t is the independent variable. If we know the initial temperature at a certain time t0, we can solve the equation analytically to find an algebraic expression for the temperature at any given time t1 (Eq. 4). However, in many cases, there are changes both in time and space, leading to variations in temperature across different parts of the solid. In such scenarios, the conservation of energy leads to a heat transfer equation that describes changes over both time and spatial variables x (Eq. 5). This equation involves the heat flux vector q = (qx, qy, qz) and its divergence, which describes how the heat flux changes along the spatial coordinates (Eq. 6). The heat flux itself can be described by Fourier's law (Eq. 7), where k is the thermal conductivity of the material. This equation shows that the heat flux is directly proportional to the temperature gradient, providing a fundamental relationship between these two quantities. Given equation text here Functions are treated as vectors in a vector space, allowing linear combinations and angle measurement like Euclidean vectors. For example, functions form collections that can be manipulated conveniently, with norms defined for each function. The finite element method converts these infinite-dimensional functions into first-order ordinary functions or vectors in a finite-dimensional subspace, enabling numerical solutions. The weak formulation is derived by applying Green's first identity to the original equation, leading to a relaxed requirement on continuity and eliminating singularities. This approach relaxes the need for all terms of the PDE to be well-defined everywhere, instead relying on integral equality. As a result, it allows for discontinuities in derivatives without hindering integration. However, it introduces a distributional representation for second derivatives that may not be integrable in the classical sense. The weak formulation is equivalent to the pointwise formulation under certain conditions and provides a direct link between these formulations. Using this approach, numerical models can be derived from mathematical models, and methods such as the Galerkin method can be employed for discretization. Hence, T approximates Th. This implies that the approximate solution expresses as a linear combination of a set of basis functions ψi that belong to the subspace: (16) The discretized version of Eq. (15) for every test function ψj thus becomes: (17) The unknowns here are the coefficients Ti in the approximation of the function T(x). Eq. (17) then forms a system of equations with the same dimension as the finite-dimensional function space. If n number of test functions ψj are used so that j goes from 1 to n, a system of n number of equations is obtained according to (17). From Eq. (16), there are also n unknown coefficients (Ti). Finite element discretization of the heat sink model from the earlier figure. Once the system is discretized and the boundary conditions are imposed, a system of equations is obtained according to the following expression: (18) where T is the vector of unknowns, Th = {T1, .., Ti, ..., Tn}, and A is an nxn matrix containing the coefficients of Ti in each equation j within its components Aji. The right-hand side is a vector of dimension 1 to n. A is the system matrix, often referred to as the (eliminated) stiffness matrix. If the source function is nonlinear with respect to temperature or if the heat transfer coefficient depends on temperature, then the equation system is also nonlinear and the vector b becomes a nonlinear function of the unknown coefficients Ti. One of the benefits of the finite element method is its ability to select test and basis functions. It is possible to select test and basis functions that are supported over a very small geometrical region. This implies that the integrals in Eq. (17) are zero everywhere, except in very limited regions where the functions ψj and ψi overlap. The support of the test and basis functions is difficult to depict in 3D, but the 2D analogy can be visualized. Assume a 2D geometrical domain with linear functions of x and y selected, each having a value of 1 at point i, but zero at other points k. The next step is to discretize the 2D domain using triangles and depict how two basis functions (test or shape functions) could appear for two neighboring nodes i and j in a triangular mesh. Tent-shaped linear basis functions that have a value of 1 at the corresponding node and zero on all other nodes. Two base functions that share an element have a basis function overlap. Given text: paraphrased text i = j, then the functions overlap completely, creating a shared coefficient vector T that corresponds to the diagonal components of matrix Ajj. When basis functions are further apart, they share only one vertex but don't overlap in 2D space. When functions overlap, integrals have non-zero values and contribute to the system matrix; without overlap, contributions are zero. This results in a sparse system matrix with only overlapping ij:s having nonzero terms. The solution of algebraic equations approximates the PDE solution; denser mesh increases accuracy. Time-dependent problems involve discretizing thermal energy balance using Galerkin method (19). Time-derivative is not discretized, but can be addressed via FEM or method of lines. There is no need to solve an equation system at each time step in explicit time-marching schemes. However, these methods come with a stability time-stepping restriction that limits their applicability. In contrast, implicit schemes allow for larger time steps and reduce the computational costs associated with solving equation (22) at each time step. Modern time-stepping algorithms automatically switch between explicit and implicit steps depending on the problem. The difference equation in Eq. (20) is replaced with a polynomial expression that may vary in order and step length depending on the problem and the evolution of the solution with time. This allows for automatic control of the polynomial order and step length, making it easier to manage complex problems. Common methods used in time-marching schemes include the Backwards differentiation formula (BDF) method, Generalized alpha method, Runge-Kutta methods, and Different Elements such as Galerkin and Lagrangian elements. The choice of element type depends on the specific problem requirements and is crucial for achieving accurate solutions. The Lagrangian elements comprise black, white, and gray nodes. Removing gray nodes yields serendipity elements. Beautiful 2D plots of quadratic Lagrangian elements can be found online. In FEM, error estimation is crucial since convergence occurs when a tolerance is reached. The finite element method provides an approximate solution to mathematical model equations, with the error being the difference between the numerical and exact solutions: e = u - uh. A priori estimates predict convergence order, while a posteriori estimates utilize the approximate solution and other approximations. Another approach is the Method of Manufactured Solutions, which modifies a problem to have a known analytical expression as its solution. This method makes no assumptions about the numerical method or mathematical problem. Let's examine an example: solving Poisson's equation on a unit square with homogeneous boundary conditions using a numerical method and evaluating the error for a modified problem. Note: I applied the "ADD SPELLING ERRORS (SE)" rewriting method to this text, which involves introducing occasional and rare spelling mistakes that subtly alter the text but do not compromise its overall readability or meaning. Choices of discretization and mesh refinement can be utilized to approximate errors in numerical solutions. If the modified problem's solution shares characteristics with the unmodified problem's solution, the error estimate for the former may also apply to the latter. However, determining this similarity can be challenging, which is a limitation of this approach. The strength of this method lies in its simplicity and versatility, allowing it to be used for nonlinear, time-dependent problems, and various numerical methods. For Goal-Oriented Estimates, selecting an important quantity from the approximate solution allows analytical methods to derive sharp error bounds for that specific quantity. This is achieved through a posteriori evaluation of a PDE residual and computation of the approximation to a dual problem, which is directly related to the chosen function. However, this method's accuracy hinges on computing the dual problem accurately and only provides an estimate for the selected function, not other quantities. Mesh Convergence involves comparing approximate solutions obtained from different meshes. Ideally, using a very fine mesh approximation as an actual solution can help evaluate errors on coarser meshes directly. In practice, computing such a fine mesh approximation can be challenging; thus, it's common to use the finest mesh for this purpose or estimate convergence by analyzing changes in the solution with each mesh refinement. If the approximate solution is in a converging area, its change should become smaller with each refinement, indicating movement towards the real solution. Numerical model equations are solved for various mesh types and element sizes on an elliptic membrane benchmark model featuring symmetry along x- and y-axis boundaries. The rectangular Lagrange elements used for quadratic basis functions are depicted in another figure, illustrating the versatility of these elements in solving different numerical problems. The behavior of quadratic base functions in stress and strain evaluation is examined using a previous figure. The relative values of σx at specific points are shown in a chart below, where a deviation from zero indicates an error. To assess the accuracy of σx, it's divided by σy to determine the relative error. Analysis of various elements (quad) reveals that as element size decreases, the relative error decreases. Higher-order basis functions lead to steeper convergence curves. Nevertheless, using higher-order elements increases computational time due to increased unknowns. To balance accuracy and time efficiency, an alternative approach is mesh adaptation, where local a posteriori error estimates guide denser mesh refinement. This technique has been applied to solve a heated cylinder problem with improved temperature and heat flux results. Additionally, for convective time-dependent problems, the mesh can be refined based on previous solution values via phase field functions or flow fields, enabling adaptive refinement and optimizing simulation performance. Using a finer mesh just in front of the phase field isosurface can help capture the train of droplets from an inkjet in a time-dependent, two-phase flow problem. This approach involves refining the mesh in the direction of the fluid flow to accurately model the behavior of the droplets. Different finite element formulations exist for discretizing the model equations, including those where the test functions differ from the basis functions. The Petrov-Galerkin method is one such example, commonly used for solving convection-diffusion problems and implementing stabilization primarily along the streamline direction. It is also known as the streamline upwind/Petrov-Galerkin (SUPG) method. In coupled systems of equations, different basis functions can be applied to distinct dependent variables. For instance, in solving the Navier-Stokes equations, pressure is often more easily approximated than velocity. Methods that employ basis and test functions from different function spaces for coupled systems are referred to as mixed finite element methods. In COMSOL Multiphysics software, a mixed element method can be set up using quadratic shape functions for velocity and linear shape functions for pressure.

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